

# $C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations.

I. Birindelli, F. Demengel

## 1 Introduction

In a recent paper Imbert and Silvestre [11] have proved that the solutions of

$$|\nabla u|^\alpha F(D^2 u) = f(x) \text{ in } \Omega \subset \mathbb{R}^N \quad (1.1)$$

when  $F$  is uniformly elliptic and  $f$  is continuous, have first derivative which are Hölder continuous when  $\alpha \geq 0$  in the interior of  $\Omega$ .

Results concerning regularity of solutions have an intrinsic interest which doesn't need to be explained. When  $\alpha = 0$ , it is known (see e.g. Evans [8], Cabré, Caffarelli [6], [5], [7]) that  $u$  is  $C^{1,\beta}$  for some  $\beta \in (0, 1)$ ; when  $F$  is concave in the Hessian the solutions are  $C^{2,\beta}$ . But for solutions of (1.1) with  $\alpha > -1$  the question of the continuity of the gradient was open and it was naturally raised, in [3]. We shall briefly recall why. The values

$$\lambda^+ = \sup\{\lambda, \exists \phi > 0 \text{ in } \Omega, |\nabla \phi|^\alpha F(D^2 \phi) + \lambda \phi^{1+\alpha} \leq 0 \text{ in } \Omega\}$$

$$\lambda^- = \sup\{\lambda, \exists \psi < 0 \text{ in } \Omega, |\nabla \psi|^\alpha F(D^2 \psi) + \lambda |\psi|^\alpha \psi \geq 0 \text{ in } \Omega\}$$

are generalized principal eigenvalues in the sense that there exists a non trivial solution to the Dirichlet problem

$$|\nabla \phi|^\alpha F(D^2 \phi) + \lambda^\pm |\phi|^\alpha \phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega,$$

with constant sign.

The main scope in [3] was to prove the simplicity of these principal eigenvalues. The difficulty comes from the fact that the strong comparison principle holds only in open subsets of  $\Omega$  where the gradient is bounded away from zero (in the viscosity sense). It is well known that the Hopf Lemma guaranties that

this is true on  $\partial\Omega$ . So that the continuity of the gradient up to the boundary implies that, in a neighborhood of it, the strong comparison principle holds, which is exactly what is needed to prove that the eigenvalues are simple.

In that same paper, we proved that if  $\alpha \in (-1, 0)$  the solutions of the Dirichlet problem associated to (1.1) are indeed  $C^{1,\beta}$  and we raised the problem of whether that regularity would hold also for  $\alpha \geq 0$  i.e. when the operator is degenerate elliptic.

[11] was a first answer in that direction. Very much inspired by that breakthrough, we wanted to complete the work, since, as we have just explained above, in order to use the regularity result in the proof of the simplicity of the eigenvalues, it is essential to prove the regularity of the derivative up to the boundary.

Recall that  $F$  is uniformly elliptic if there exists  $\Lambda \geq \lambda > 0$  such that for any symmetric matrix  $M$

$$\mathcal{M}_{\lambda,\Lambda}^-(M) \leq F(M) \leq \mathcal{M}_{\lambda,\Lambda}^+(M) \quad (1.2)$$

here,  $\mathcal{M}_{\lambda,\Lambda}^-(M) = \lambda \operatorname{tr}(M^+) + \Lambda \operatorname{tr}(M^-)$  and  $\mathcal{M}_{\lambda,\Lambda}^+(M) = \lambda \operatorname{tr}(M^-) + \Lambda \operatorname{tr}(M^+)$ . In the rest of the paper we shall drop the indices  $\lambda$  and  $\Lambda$  of the Pucci operators.

We now state the regularity's result we prove in this paper.

**Theorem 1.1** *Suppose that  $\Omega$  is a bounded  $C^2$  domain of  $\mathbb{R}^N$  and  $\alpha \geq 0$ . Suppose that  $F$  is uniformly elliptic, that  $h$  is a continuous function such that  $(h(x) - h(y)) \cdot (x - y) \leq 0$ . Let  $f \in C(\overline{\Omega})$  and  $\varphi \in C^{1,\beta_0}(\partial\Omega)$ . For any  $u$ , viscosity solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

there exist  $\beta = \beta(\lambda, \Lambda, |f|_\infty, N, \Omega, |h|_\infty, \beta_0)$  and  $C = C(\beta)$  such that

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \leq C \left( \|\varphi\|_{C^{1,\beta_0}(\partial\Omega)} + |u|_{L^\infty(\Omega)} + |f|_{L^\infty(\Omega)}^{\frac{1}{1+\alpha}} \right).$$

For radial solutions, and a more general class of operators, this was proved in [4], with the optimal Hölder's coefficient  $\beta = \frac{1}{1+\alpha}$ .

The novelty with respect to the paper of Imbert and Silvestre is two folded, on one hand we have added the lower order term  $h(x) \cdot \nabla u |\nabla u|^\alpha$ , and on the other hand we go all the way to the boundary. The proof follows the scheme of

the one in [11], but requires new tools and new ideas. In particular in section 2, we give some a priori Lipschitz and Hölder estimates in the presence of boundary condition on one part of the boundary. These are important because the proof of Theorem 1.1 requires that sequence of bounded solutions do converge to a solution of a limit equation. The main tool remains an "improvement of flatness lemma". The reader will see that the presence of the lower order term and of the boundary term requires some new idea in order to complete the proof.

For completeness sake, let us now write down the theorem concerned with the simplicity of the principal eigenvalues.

**Theorem 1.2** *Suppose that  $\Omega$  is a bounded  $\mathcal{C}^2$  domain, such that  $\partial\Omega$  is connected. Then let  $\varphi$  and  $\psi$  be two positive eigenfunctions for the principal eigenvalue  $\bar{\lambda}$ , i.e. they are both solutions of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) + \bar{\lambda} u^{1+\alpha} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*Then there exists  $t > 0$  such that  $\varphi = t\psi$ .*

*Of course the same result holds for  $\underline{\lambda}$  and negative eigenfunctions.*

We will not give the proof of this Theorem, since the proof given in [3] in the case  $\alpha \leq 0$  can be extended to the case  $\alpha > 0$  as soon as the eigenfunctions are  $\mathcal{C}^1$  near the boundary.

## 2 Local Hölder and Lipschitz estimates up to the boundary.

Throughout the paper, the notation  $B_r(x)$  indicates the ball of radius  $r$  and center  $x$ , the center may be dropped if no ambiguity arise.

It is a classical fact that in order to prove that  $u$  is  $\mathcal{C}^{1,\beta}$  at  $x_o$ , it is enough to prove that there exists some constant  $C$  such that, for all  $r < 1$ , there exists  $p_r$ , such that  $\text{osc}_{B_r(x_o)}(u(x) - p_r \cdot x) \leq Cr^{1+\beta}$ .

Furthermore,  $u$  is  $\mathcal{C}^{1,\beta}$  in some bounded open set  $B$  if there exists a constant  $C_\beta$  such that for all  $x \in B$  and  $r < 1$ , there exists  $p_{r,x}$  such that

$$\text{osc}_{B_r(x)}(u(y) - p_{r,x} \cdot y) \leq C_\beta r^{1+\beta}.$$

This will be used in the whole paper.

We begin by stating the following comparison theorem which will be employed several times.

**Theorem 2.1** [1] Suppose that  $\Omega$  is a bounded open set in  $\mathbb{R}^N$ . Let  $u$  and  $v$  be in  $\mathcal{C}(\overline{\Omega})$  and respectively solutions of

$$|\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) \leq f \quad \text{in } \Omega$$

and

$$|\nabla v|^\alpha (F(D^2v) + h(x) \cdot \nabla v) \geq g \quad \text{in } \Omega$$

with  $f$  and  $g$  continuous and bounded such that  $f < g$ .

If  $u \geq v$  on  $\partial\Omega$  then  $u \geq v$  in  $\Omega$ .

In order to prove Hölder and Lipschitz estimates we fixe a few notations concerning  $\Omega$  and  $F$ . We suppose, without loss of generality, that at  $0 \in \partial\Omega$ , the interior normal is  $e_N$ . By the implicit function theorem, there exist a ball  $B = B_R(0)$  in  $\mathbb{R}^N$ , a ball  $B' = B'_R(0)$  of  $\mathbb{R}^{N-1}$  and  $a \in \mathcal{C}^2(B'_R(0))$ , such that  $a(0) = 0$ ,  $\nabla a(0) = 0$  and, for  $y = (y', y_N)$ ,

$$\Omega \cap B = \{y_N > a(y'), y' \in B'\}, \text{ and } \partial\Omega \cap B = \{y_N = a(y'), y' \in B'\}.$$

We shall also act as if  $F$  be positively homogenous of degree 1 i.e. such that for any  $t > 0$ ,  $F(tM) = tF(M)$ . Observe though that, if this doesn't hold, when it occurs, we replace in the computations  $F(M)$  by  $G_t(M) = t^{-1}F(tM)$ ; this operator satisfies (1.2) with the same constant than  $F$  and the results are unchanged.

In the lemma below we have supposed, for simplicity, that  $B$  is the unit ball centered at the origin.

**Lemma 2.2** Let  $\varphi$  be a Hölder continuous function. Let  $a \in \mathcal{C}^2(B')$  such that  $a(0) = 0$  and  $\nabla a(0) = 0$ . Let  $d$  be the distance to the hypersurface  $\{y_N = a(y')\}$ .

Then, for all  $r < 1$  and for all  $\gamma < 1$ , there exists  $\delta_o = \delta_o(|f|_\infty, \gamma, r, \lambda, \Lambda, |h|_\infty, \Omega, \varphi)$ , such that for all  $\delta < \delta_o$  any  $u$ ,  $|u|_\infty \leq 1$ , solution of

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(y) \cdot \nabla u) = f & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases} \quad (2.1)$$

satisfies  $|u(y', y_N) - \varphi(y')| \leq \frac{6}{\delta} \frac{d(y)}{1+d(y)^\gamma}$  in  $B_r \cap \{y_N > a(y')\}$ .

*Proof.* We write the details of the proof for  $\varphi = 0$ . The changes to bring in the case where  $\varphi \neq 0$  will be given at the end of the proof, the detailed calculation being left to the reader.

It is sufficient to consider the set where  $d(y) < \delta$  since the assumption  $|u|_\infty \leq 1$  implies the result elsewhere.

We begin to choose  $\delta < \delta_1$ , such that on  $d(y) < \delta_1$  the distance is  $\mathcal{C}^2$  and satisfies  $|D^2d| \leq C_1$ . We shall also choose later  $\delta$  smaller in function of  $(\lambda, \Lambda, |f|_\infty, |h|_\infty, N)$ .

Let

$$w(y) = \begin{cases} \frac{2}{\delta} \frac{d(y)}{1+d^\gamma(y)} & \text{for } |y| < r \\ \frac{2}{\delta} \frac{d(y)}{1+d^\gamma(y)} + \frac{1}{(1-r)^3} (|y| - r)^3 & \text{for } |y| > r. \end{cases}$$

We first remark that

$$w \geq u \text{ on } \partial(B \cap \{y_N > a(y')\} \cap \{d(y) < \delta\}).$$

Indeed, let us observe that on  $\{d(y) = \delta\}$ ,  $w \geq \frac{2}{\delta} \frac{\delta}{1+\delta^\gamma} \geq 1 \geq u$ . On  $\{|y| = 1\} \cap \{d(y) < \delta\}$ ,  $w \geq \frac{1}{(1-r)^3} (1-r)^3 \geq u$ . On  $B \cap \{y_N = a(y')\}$ ,  $w \geq 0 = u$ .

We need to choose  $\delta$  small enough that  $w$  satisfies

$$|\nabla w|^\alpha (\mathcal{M}^+(D^2w) + h(y) \cdot \nabla w) < -|f|_\infty, \text{ in } B \cap \{y_N > a(y')\} \cap \{d(y) < \delta\}. \quad (2.2)$$

For that aim, we compute

$$\nabla w = \begin{cases} \frac{2}{\delta} \frac{1+(1-\gamma)d^\gamma}{(1+d^\gamma)^2} \nabla d & \text{when } |y| < r \\ \frac{2}{\delta} \frac{1+(1-\gamma)d^\gamma}{(1+d^\gamma)^2} \nabla d + \frac{y}{|y|} \frac{3}{(1-r)^3} (|y| - r)^2 & \text{if } |y| > r. \end{cases}$$

Note that  $|\nabla w| \geq \frac{1}{4\delta}$  as soon as  $\delta \leq \frac{1-r}{12}$ . By construction  $w$  is  $\mathcal{C}^2$  and

$$D^2w = \begin{cases} -(\frac{\gamma d^{\gamma-1}}{2\delta}) \frac{(1+\gamma)+(1-\gamma)d^\gamma}{(1+d^\gamma)^3} \nabla d \otimes \nabla d + \frac{2}{\delta} \frac{1+(1-\gamma)d^\gamma}{(1+d^\gamma)^2} D^2d & \text{if } |y| < r \\ -(\frac{\gamma d^{\gamma-1}}{2\delta}) \frac{(1+\gamma)+(1-\gamma)d^\gamma}{(1+d^\gamma)^3} \nabla d \otimes \nabla d + \frac{2}{\delta} \frac{1+(1-\gamma)d^\gamma}{(1+d^\gamma)^2} D^2d + H(y) & \text{if } |y| > r \end{cases}$$

where  $\|H(y)\| \leq \frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)}$ .

We now choose  $\delta$  small enough in order that  $\delta < \delta_1$  and

$$\lambda(\gamma\delta^{\gamma-2}) \frac{(1+\gamma)}{(1+\delta^\gamma)^3} > 2\Lambda \left( \frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)} + \frac{2C_1}{\delta} \right) + \frac{4|h|_\infty}{\delta}$$

and also such that

$$\frac{\lambda}{2^{2+\alpha}} (\gamma\delta^{\gamma-(2+\alpha)}) \frac{(1+\gamma)}{(1+\delta^\gamma)^3} > \|f\|_\infty.$$

These choices of  $\delta$ , and standard computations that use (1.2) imply that  $w$  satisfies (2.2). The comparison principle gives that  $u \leq w$  in  $B \cap \{y_N > a(y')\} \cap \{d(y) < \delta\}$ .

Furthermore the desired lower bound on  $u$  is easily deduced by considering  $-w$  in place of  $w$  in the previous computations and restricting to  $B_r \cap \{y_N > a(y')\}$ . This ends the proof of Lemma 2.2.

When  $\varphi \not\equiv 0$ , we extend  $\varphi$  as a solution  $\psi$  of

$$\begin{cases} \mathcal{M}^+(D^2\psi) = 0 & \text{in } B \cap \{y_N > a(y')\} \\ \psi = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases}$$

and  $\psi$  is  $\mathcal{C}^{1,\beta_0}(B \cap \{y_N \geq a(y')\}) \cap \mathcal{C}^2(B \cap \{y_N > a(y')\})$ . Furthermore, we can choose  $\psi$  such that  $|\psi|_\infty \leq |\varphi|_\infty \leq 1$ ,  $|\nabla\psi|_\infty \leq c|\nabla\varphi|_\infty$ , for some constant which depends on  $\lambda, \Lambda, N, \Omega$  [7].

We now define

$$w(y) = \begin{cases} \frac{8}{\delta} \frac{d(y)}{1+d^\gamma(y)} + \psi(y) & \text{for } |y| < r \\ \frac{8}{\delta} \frac{d(y)}{1+d^\gamma(y)} + \frac{1}{(1-r)^3}(|y| - r)^3 + \psi(y) & \text{for } |y| > r. \end{cases}$$

Similar computations imply that

$$w \geq u \text{ on } \partial(B \cap \{y_N > a(y')\}) \cap \{d(y) < \delta\}.$$

Furthermore choosing  $\delta$  small enough, we can ensure that

$$|\nabla w|^\alpha (\mathcal{M}(D^2w) + h(x) \cdot \nabla w) < -|f|_\infty.$$

For the lower bound, we replace  $w$  by  $2\psi - w$ .

Using this estimate together with an argument due to Ishii and Lions, [12], one finally gets the Hölder regularity of the solution, which can be stated as follows with the same hypothesis on  $a$ ,  $B$  and  $f$  as above :

**Proposition 2.3** *Let  $\varphi$  be a Lipschitz continuous function. Suppose that  $u$  satisfies (2.1).*

*For all  $r < 1$ , and for all  $\gamma$ ,  $u$  is  $\gamma$  Hölder continuous on  $B_r \cap \{y_N > a(y')\}$ , with some Hölder's constant depending on  $(r, \lambda, \Lambda, a, N, |f|_\infty, |h|_\infty, \text{Lip}\varphi)$ .*

**Remark 2.4** *In the absence of boundary conditions, the solutions are Hölder continuous inside  $B_r$  for any  $r$  such that  $B_r \subset\subset B$ . We do not give the proof which follows the lines in the proof below, it is sufficient to cancel in it the dependence on  $\varphi$ . This will be used in the proof of the interior improvement of flatness lemma with additional lower terms.*

*Proof of Proposition 2.3.* We use both some arguments in [1] and [11]. Let  $1 > r' > r$ . Without loss of generality we can suppose that  $\text{osc } u \leq 1$ . Let  $x_o \in B_r \cap \{y_N > a(y')\}$  and  $\Phi$  defined as

$$\Phi(x, y) = u(x) - u(y) - M|x - y|^\gamma - L|x - x_o|^2 - L|y - x_o|^2.$$

The scope is to prove that for  $L$  and  $M$  independent of  $x_o$ , chosen large enough,

$$\Phi(x, y) \leq 0 \text{ on } B_{r'} \cap \{y_N > a(y')\}. \quad (2.3)$$

This will imply that  $u$  is  $\gamma$ -Hölder continuous on  $B_r \cap \{y_N > a(y')\}$  by taking  $x = x_o$ .

To prove (2.3), we begin to observe that it is true on the boundary. First we suppose that  $y_N = a(y')$ . According to Lemma 2.2 there exists  $M_o$  such that for  $x \in B_{r'} \cap \{y_N > a(y')\}$ ,

$$|u(x) - \varphi(x')| \leq M_o d(x, \partial\Omega).$$

Then, using  $|x' - y'| \leq |x - y|$ , one has

$$\begin{aligned} |u(x', x_N) - u(y', a(y'))| &\leq |u(x', x_N) - u(x', a(x'))| + |u(x', a(x')) - u(y', a(y'))| \\ &\leq M_o d(x, \partial\Omega) + \text{Lip}_\varphi |x' - y'| \\ &\leq M_o |x - (y', a(y'))| + \text{Lip}_\varphi |x - (y', a(y'))|. \end{aligned}$$

So, if  $M$  is chosen greater than  $M_o + \text{Lip}_\varphi$ , then we have obtained that  $\Phi \leq 0$  on  $B_{r'} \cap \{y_N = a(y')\}$ .

On the rest of the boundary it is enough to choose  $L > \frac{4}{(r'-r)^2}$  and to recall that the oscillation of  $u$  is bounded by 1.

In the sequel we will choose  $M$  large in order that  $\frac{L}{M} = o(1)$ .

Suppose by contradiction that  $\Phi(x, y) > 0$  for some  $(x, y) \in B_{r'} \cap \{y_N > a(y')\}$ . Then there exists  $(\bar{x}, \bar{y})$  such that

$$\Phi(\bar{x}, \bar{y}) = \sup_{\overline{B_{r'}}} (\Phi(x, y)) > 0.$$

Clearly  $\bar{x} \neq \bar{y}$ . Furthermore the hypothesis on  $L$  forces  $\bar{x}$  and  $\bar{y}$  to be in  $B_{\frac{r'+r}{2}} \cap \{y_N > a(y')\}$ . Then, for all  $\epsilon > 0$  small depending on the norm of  $Q := D^2(M|x - y|^\gamma)$ , using Ishii's Lemma [10], there exist  $X$  and  $Y$  such that

$$(\gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma-2} + 2L(\bar{x} - x_o), X) \in J^{2,+}u^*(\bar{x})$$

$$(\gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma-2} - 2L(\bar{y} - x_o), -Y) \in J^{2,-}u_*(\bar{y})$$

with

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix} + (2L + \epsilon) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

In the sequel, since we assumed that  $\frac{L}{M} = o(1)$ , one also has  $\frac{L+\epsilon}{M} = o(1)$  and then we drop  $\epsilon$  for simplicity.

Let us denote  $q_x = \gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma-2} + 2L(\bar{x} - x_o)$ , and  $q_y = \gamma M(\bar{x} - \bar{y})|\bar{x} - \bar{y}|^{\gamma-2} - 2L(\bar{y} - x_o)$ . By the choice of  $M$  in function of  $L$ ,

$$2\gamma M|\bar{x} - \bar{y}|^{\gamma-1} \geq (|q_x|, |q_y|) \geq \frac{1}{2}\gamma M|\bar{x} - \bar{y}|^{\gamma-1},$$

and since  $|q_x - q_y| \leq 4L$ , by the mean value's theorem and using a constant  $\kappa < \alpha$  if  $\alpha < 1$  and  $\kappa = 1$  if  $\alpha > 1$ :

$$\begin{aligned} ||q_x|^\alpha - |q_y|^\alpha| &\leq \alpha |q_x - q_y|^\kappa 2^{|\alpha-1|} |\gamma M|\bar{x} - \bar{y}|^{\gamma-1}|^{1-\kappa} |\gamma M|\bar{x} - \bar{y}|^{\gamma-1}|^{\alpha-1} \\ &\leq C(M\gamma|\bar{x} - \bar{y}|^{\gamma-1})^{\alpha-\kappa} \left(\frac{L}{M}\right)^\kappa = o((M|\bar{x} - \bar{y}|^{(\gamma-1)})^{(\alpha-\kappa)}). \end{aligned}$$

We now treat the terms concerning the second order derivative. The previous inequalities can also be written as

$$\begin{pmatrix} X - 2LI & 0 \\ 0 & Y - 2LI \end{pmatrix} \leq \begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix}.$$

We prove in what follows that, for some constant which can vary from one line to another,  $L = o(|\text{tr}(X + Y)|)$ , and that there exist constant  $c$  and  $C$  such that  $|X|, |Y| \leq C|\text{tr}(X + Y)|$  and

$$cM|\bar{x} - \bar{y}|^{\gamma-2} \leq |\text{tr}(X + Y)| \leq CM|\bar{x} - \bar{y}|^{\gamma-2}.$$

Indeed, let

$$P := \frac{(\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y})}{|\bar{x} - \bar{y}|^2} \leq I.$$



Using  $-(X + Y) \geq 0$  and  $(I - P) \geq 0$  and the properties of the symmetric matrices one has

$$\operatorname{tr}(X + Y - 4L) \leq \operatorname{tr}(P(X + Y - 4L)).$$

Remarking in addition that  $X + Y - 4L \leq 4Q$ , one sees that  $\operatorname{tr}(X + Y - 4L) \leq 4\operatorname{tr}(PQ)$ . But  $\operatorname{tr}(PQ) = \gamma M(\gamma - 1)|\bar{x} - \bar{y}|^{\gamma-2} < 0$ , hence

$$|\operatorname{tr}(X + Y - 4L)| \geq 4\gamma M(1 - \gamma)|\bar{x} - \bar{y}|^{\gamma-2}. \quad (2.4)$$

Furthermore by Lemma III.1 of [12] there exists a universal constant  $C$  such that

$$\begin{aligned} |X|, |Y|, |X - 2L|, |Y - 2L| &\leq C(|\operatorname{tr}(X + Y - 4L)| + |Q|^{\frac{1}{2}}|\operatorname{tr}(X + Y - 4L)|^{\frac{1}{2}}) \\ &\leq C|\operatorname{tr}(X + Y - 4L)| \\ &\leq C|\operatorname{tr}(X + Y)| \end{aligned}$$

since  $|Q|$  and  $|\operatorname{tr}(X + Y - 4L)|$  are of the same order, and  $\frac{L}{M} = o(1)$ . This will yield the required estimates.

For some positive constants  $c_2, c_3$ , since  $u$  is both a sub- and a super-solution of (2.1), using the uniform ellipticity of  $F$  and the assumptions on  $h$ :

$$\begin{aligned} f(\bar{x}) &\leq |q_x|^\alpha (F(X) + h(\bar{x}) \cdot q_x) \\ &\leq |q_y|^\alpha (F(X) + h(\bar{x}) \cdot q_x) \\ &\quad + o(M\gamma|\bar{x} - \bar{y}|^{\gamma-1})^{\alpha-\kappa} (\Lambda|X| + |h|_\infty(\gamma M|\bar{x} - \bar{y}|^{\gamma-1} + 2L)) \\ &\leq |q_y|^\alpha (F(Y) + h(\bar{y}) \cdot q_x + 4|h|_\infty L) + \\ &\quad + |q_y|^\alpha \operatorname{tr}(X + Y) + o(M^{1+\alpha}|\bar{x} - \bar{y}|^{(\gamma-1)\alpha+\gamma-2}) \\ &\leq M^\alpha c_2 |\bar{x} - \bar{y}|^{(\gamma-1)\alpha} \operatorname{tr}(X + Y) + o(M^{1+\alpha}|\bar{x} - \bar{y}|^{(\gamma-1)\alpha+\gamma-2}) + f(\bar{y}) \\ &\leq -c_3 M^{1+\alpha} |\bar{x} - \bar{y}|^{(\gamma-1)\alpha+\gamma-2} + f(\bar{y}). \end{aligned}$$

This is clearly false as soon as  $M$  is large enough and it ends the proof.

Compactness near the boundary is a natural consequence of Proposition 2.3.

**Corollary 2.5** *Suppose that  $(u_n)$  is a bounded sequence of continuous functions which satisfy*

$$\begin{cases} |\nabla u_n|^\alpha (F(D^2 u_n) + h(y) \cdot \nabla u_n) = f_n & \text{in } B \cap \{y_N > a(y')\} \\ u_n = \varphi & \text{on } B \cap \{y_N = a(y')\} \end{cases}$$

and suppose that  $(f_n)$  converges simply to some continuous function  $f$ . Then for all  $r < 1$ , one can extract from  $(u_n)$  a subsequence which converges uniformly, on  $\overline{B_r} \cap \{y_N > a(y')\}$ , towards a solution of

$$\begin{cases} |\nabla u|^\alpha (F(D^2 u) + h(y) \cdot \nabla u) = f & \text{in } B \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\}. \end{cases}$$

**Remark 2.6** *In the absence of boundary conditions, the analogous result holds, in the sense that one can extract from  $(u_n)_n$  a subsequence which converges uniformly on every  $B_r \subset\subset B$  to  $u$  a solution of*

$$|\nabla u|^\alpha (F(D^2 u) + h(y) \cdot \nabla u) = f \text{ in } B.$$

When we shall treat, in the improvement of flatness lemma up to the boundary, the case where the boundary is locally straight, we shall need the following Lipschitz estimate's near the boundary for some different but related equation.

**Proposition 2.7** *(Lipschitz estimates for large  $p$ 's) Let  $\varphi$  be a Lipschitz continuous function. Assume that  $u$  solves*

$$\begin{cases} |pe_N + \nabla u|^\alpha (F(D^2 u) + h(y) \cdot \nabla u) = f & \text{in } B(x) \cap \{y_N > 0\} \\ u = \varphi & \text{on } \{y_N = 0\} \cap B_1(x) \end{cases}$$

with  $|u|_{L^\infty(B_1(x) \cap \{y_N > 0\})} \leq 1$  and  $\|f\|_{L^\infty(B(x) \cap \{y_N > 0\})} \leq \epsilon_o < 1$ . Then, for all  $r < 1$ , there exists  $b_o$  depending on  $(\lambda, \Lambda, N, \alpha, r, \epsilon_o, \text{Lip}\varphi)$ , such that if  $|p| > \frac{1}{b_o}$ ,  $u$  is Lipschitz continuous in  $B_r(x) \cap \{y_N > 0\}$  with some Lipschitz constant depending on  $(\lambda, \Lambda, N, \alpha, r, \epsilon_o, \text{Lip}\varphi)$ .

**Remark 2.8** *In the absence of boundary conditions, the solutions are Lipschitz with Lipschitz constant independent of  $p$ , inside  $B_r$ , for any  $r$  such that  $B_r \subset\subset B$ .*

*This will be used in the proof of the interior improvement of flatness lemma with lower order terms.*

This Proposition is a consequence of the following

**Lemma 2.9** *For all  $\gamma < 1$ , for all  $r < 1$ , there exists  $\delta = \delta(|f|_\infty, \lambda, \Lambda, r)$ , such that for  $b < \frac{\delta}{4}$ , any solution  $u$  of*

$$\begin{cases} |e_N + b\nabla u|^\alpha (F(D^2u) + h(y) \cdot \nabla u) = f & \text{in } B(x) \cap \{y_N > 0\} \\ u = \varphi & \text{on } B(x) \cap \{y_N = 0\}. \end{cases}$$

*such that  $\text{osc}(u) \leq 1$  satisfies  $|u(y', y_N) - \varphi(y')| \leq \frac{2}{\delta} \frac{y_N}{1+y_N^\gamma}$  in  $B_r \cap \{y_N > 0\}$ .*

*Proof of Lemma 2.9.* Suppose for simplicity that  $\varphi = 0$ . If  $b = 0$  the result is known by properties of solutions of  $F(D^2u) + h(x) \cdot \nabla u = f$  which are zero on the boundary. So we assume in what follows that  $b \neq 0$ .

We act as in Lemma 2.2, where the distance is replaced by  $y_N$  so we consider

$$w(y) = \begin{cases} \frac{2}{\delta} \frac{y_N}{1+y_N^\gamma} & \text{for } y_N < \delta, |y'| < r \\ \frac{2}{\delta} \frac{y_N}{1+y_N^\gamma} + \frac{1}{(1-r)^3} (|y'| - r)^3 & \text{for } y_N < \delta, |y'| > r. \end{cases}$$

Similarly to the proof of Lemma 2.2, it is sufficient to consider the set where  $y_N < \delta$ , since the assumption  $|u|_\infty \leq 1$  implies the result elsewhere. Furthermore we only prove that  $u \leq w$ , the desired lower bound can be obtained by considering  $-w$  in place of  $w$ .

In order for  $w$  to satisfy

$$|e_N + b\nabla w|^\alpha (\mathcal{M}^+(D^2w) + h(y) \cdot \nabla w) \leq -|f|_\infty, \text{ in } B,$$

it is sufficient to choose  $\delta$  such that

$$\left(\frac{1}{2}\right)^\alpha \lambda \gamma \delta^{\gamma-2} (1-\gamma) \frac{1}{(1+\delta^\gamma)^3} > |f|_\infty + 2\Lambda \left( \frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)} + \frac{4|h|_\infty}{\delta} \right),$$

$b < \frac{\delta}{4}$ , and recall that  $|\nabla w| \leq \frac{2}{\delta}$ . Furthermore  $w \geq u$  on  $\partial(B^+ \cap \{0 < x_N < \delta\})$ .

The comparison principle in Theorem 2.1, between the functions  $x \mapsto x_N + bw(x)$  and  $x \mapsto x_N + bu(x)$ , as well as  $b \neq 0$ , implies that  $u \leq w$  in  $B \cap \{y_N > 0\}$ . Finally the desired estimate is obtained in  $\{|y'| < r, y_N > 0\}$ .

In the case  $\varphi \neq 0$ , we take the function  $w$  as in the proof of Lemma 2.2, with  $d$  replaced by  $y_N$ . Requiring sufficient restriction on the smallness of  $\delta$  give the result.

We are now ready to give the

*Proof of Proposition 2.7.* We act as in [11]. We rewrite the equation as

$$|e_N + bDu|^\alpha (F(D^2u) + h(y) \cdot \nabla u) = \tilde{f}$$

with  $b = \frac{1}{p}$  and  $\tilde{f} = |p|^{-\alpha} f$ .

Choose first  $\delta$  small enough in order that

$$\begin{aligned} \left(\frac{1}{2}\right)^\alpha \lambda(\gamma\delta^{\gamma-2}) \frac{(1+\gamma)}{2(1+\delta\gamma)^3} &> |f|_\infty + 2\Lambda\left(\frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)} + \frac{4|h|_\infty}{\delta}\right) \\ &> |\tilde{f}|_\infty + 2\Lambda\left(\frac{6}{(1-r)^2} + \frac{3(N-1)}{r(1-r)} + \frac{4|h|_\infty}{\delta}\right). \end{aligned}$$

and such that  $b < \frac{\delta}{4}$ . Let us note that this implies that  $b$  is small enough depending on  $(\lambda, \Lambda, N, \alpha, r, \epsilon_o, |h|_\infty, \text{Lip}\varphi)$ .

Let  $r < r' < 1$  and let  $x_o \in B_{r'}(x) \cap \{y_N > 0\}$ ,  $L_2 = \frac{4}{(r'-r)^2}$ ,

$$\psi(z, y) = u(z) - u(y) - L_1\omega(|z - y|) - L_2|z - x_o|^2 - L_2|y - x_o|^2$$

where  $\omega(s) = s - \omega_o s^{\frac{3}{2}}$  if  $s \leq s_o = \left(\frac{2}{3\omega_o}\right)^2$  and  $\omega(s) = \omega(s_o)$  if  $s \geq s_o$ . We also require  $L_1 > \frac{2}{\delta} + \text{Lip}\varphi$ .

If we prove that  $\psi(z, y) \leq 0$  in  $B_{r'}$ , since  $L_1$  is independent of  $x_o$ , by choosing  $z = x_o$  one gets

$$u(z) - u(y) \leq L_1|z - y| + L_2|z - y|^2$$

which implies the desired result when  $z \in B_r(x)$  or  $y \in B_r(x)$ .

We begin to observe that if the supremum is achieved on  $\overline{B_r(x)}$  and if  $(\bar{x}, \bar{y})$  is a point where the supremum is achieved, then, with our choice of  $L_1$ , neither  $\bar{x}$  nor  $\bar{y}$  can belong to the part  $\{z_N = 0\}$  according to Lemma 2.9. The rest of the proof is as in [11], see also the proof of Proposition 2.3.

As a corollary of this Lemma one has the following compactness result

**Corollary 2.10** *Let  $\varphi$  be a Lipschitz continuous function. Suppose that  $(u_n)$  is a sequence of continuous viscosity solutions of*

$$\begin{cases} |e_N + b_n \nabla u_n|^\alpha (F(D^2 u_n) + h \cdot \nabla u_n) = f_n & \text{in } B, \\ u_n = \varphi & \text{on } B \cap \{y_N = a(y')\}. \end{cases}$$

where  $b_n \leq b_o$ ,  $b_o$  is given above in Proposition 2.7. Suppose that  $f_n$  converges simply to some function  $f$  in  $\Omega$ . Then for all  $r < 1$ , one can extract from

$(u_n, b_n)$  a subsequence which converges uniformly on  $\overline{B_r \cap \{y_N > a(y')\}} \times \mathbb{R}$ , and the limit  $(u, \bar{b})$  satisfies

$$\begin{cases} |e_N + \bar{b}\nabla u|^\alpha (F(D^2u) + h \cdot \nabla u) = f & \text{in } B \\ u = \varphi & \text{on } B \cap \{y_N = a(y')\}. \end{cases}$$

**Remark 2.11** *In the absence of boundary conditions, the conclusion is that the sequence  $(u_n)$  contains a subsequence which converges locally uniformly and up to a constant toward a solution of*

$$|e_N + \bar{b}\nabla u|^\alpha (F(D^2u) + h \cdot \nabla u) = f \text{ in } B.$$

### 3 Proof of Theorem 1.1.

In fact Theorem 1.1 is an immediate consequence of the following local result up to the boundary together with some argument of finite covering:

**Theorem 3.1** *Suppose that  $F$ ,  $h$  and  $f$  are as in Theorem 1.1. Let  $B$  be an open set in  $\mathbb{R}^N$  and let  $a$  be a  $\mathcal{C}^2$  function defined on  $B \cap \mathbb{R}^{N-1} \times \{0\}$  with  $a(0) = 0$ ,  $\nabla a(0) = 0$ . There exists  $\beta$  such that for any  $u$  solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2u) + h(x) \cdot \nabla u) = f & \text{in } B \cap \{x_N > a(x')\} \\ u = \varphi & \text{on } B \cap \{x_N = a(x')\}, \end{cases}$$

$u$  is  $\mathcal{C}^{1,\beta}(\tilde{B} \cap \{x_N > a(x')\})$  for any  $\tilde{B} \subset\subset B$ .

Theorem 3.1 is proved via the following two "improvement of flatness" lemma and their consequences.

**Lemma 3.2** *There exist  $\epsilon_o \in ]0, 1[$  and  $\rho \in ]0, 1[$  depending on  $(\alpha, |h|_\infty, \lambda, \Lambda, N)$  such that for any  $p \in \mathbb{R}^N$  and for any viscosity solution  $u$  of*

$$|p + \nabla u|^\alpha (F(D^2u) + h(y) \cdot (\nabla u + p)) = f \text{ in } B_1$$

*such that  $\text{osc}_{B_1} u \leq 1$  and  $\|f\|_{L^\infty(B_1)} \leq \epsilon_o$ , there exists  $p' \in \mathbb{R}^N$  such that*

$$\text{osc}_{B_\rho}(u - p' \cdot x) \leq \frac{1}{2}\rho.$$

and

**Lemma 3.3** *For any  $a \in \mathcal{C}^2$ , such that  $a(0) = 0$  and  $\nabla a(0) = 0$ , there exist  $\epsilon_o > 0$  and  $\rho$  which depend on  $(\alpha, \lambda, \Lambda, N, |D^2 a|_\infty, |h|_\infty, |\varphi|_{\mathcal{C}^{1,\beta}})$  such that for any  $p \in \mathbb{R}^N$  and  $u$  a viscosity solution of*

$$\begin{cases} |p + \nabla u|^\alpha (F(D^2 u) + h(y) \cdot (\nabla u + p)) = f & \text{in } B \cap \{y_N > a(y')\} \\ u + p \cdot y = \varphi & \text{on } \{y_N = a(y')\} \cap B. \end{cases}$$

*Then for all  $x \in B$  such that  $B_1(x) \subset B$ ,  $\text{osc}_{B_1(x) \cap \{y_N > a(y')\}} u \leq 1$ , and  $|f|_{L^\infty(B_1(x) \cap \{y_N > a(y')\})} \leq \epsilon_o$ , there exists  $q_{x,\rho} \in \mathbb{R}^N$  such that*

$$\text{osc}_{B_\rho(x) \cap \{y_N > a(y')\}} (u(y) - q_{x,\rho} \cdot y) \leq \frac{\rho}{2}.$$

Suppose that these Lemmata have been proved and let us derive the following one.

**Lemma 3.4** *Suppose that  $\rho$  and  $\epsilon_o \in [0, 1]$  are as in lemma 3.3 and suppose that  $u$  is a viscosity solution of*

$$\begin{cases} |\nabla u|^\alpha (F(D^2 u) + h(y) \cdot \nabla u) = f & \text{in } B_1(x) \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B_1(x) \cap \{y_N = a(y')\} \end{cases} \quad (3.1)$$

*with  $\text{osc } u \leq 1$  and  $\|f\|_\infty \leq \epsilon_o$ , then, there exists  $\beta \in ]0, 1[$ , such that for all  $k$  and for all  $x \in \Omega$  there exists  $p_k \in \mathbb{R}^N$  such that*

$$\text{osc}_{B_{r^k}(x) \cap \{y_N > a(y')\}} (u(y) - p_k \cdot y) \leq r_k^{1+\alpha} \equiv \rho^{k(1+\alpha)}.$$

**Remark 3.5** *Of course, the interior regularity becomes :*

*Suppose that  $\rho$  and  $\epsilon_o \in [0, 1]$  are as in lemma 3.2 and suppose that  $u$  is a viscosity solution of*

$$|\nabla u|^\alpha (F(D^2 u) + h(\cdot) \cdot \nabla u) = f \text{ in } B_1$$

*with  $\text{osc } u \leq 1$  and  $\|f\|_\infty \leq \epsilon_o$ , then there exists  $\beta \in ]0, 1[$  such that for all  $k$  there exists  $p_k \in \mathbb{R}^N$  such that*

$$\text{osc}_{B_{r^k}} (u(x) - p_k \cdot x) \leq r_k^{1+\alpha} \equiv \rho^{k(1+\alpha)}.$$

*We will not give the proof, the changes to bring to the proof below being obvious.*

*Proof of Lemma 3.4.* As in [11] we use a recursive argument.

We first remark that one can assume that  $\varphi(x') = \partial_i \varphi(x') = 0$  for  $i = 1, \dots, N-1$ .

Indeed, let  $u$  be a solution of (3.1). Let  $v(y) = u(y) - u(x) - \nabla \varphi(x') \cdot (y' - x')$  which satisfies

$$\begin{cases} |\nabla v + q|^\alpha (F(D^2 v) + h(y) \cdot (\nabla v + q)) = f & \text{in } B_1(x) \cap \{y_N > a(y')\} \\ v(y', a(y')) = \phi(y') & \text{on } B_1(x) \cap \{y_N = a(y')\} \end{cases}$$

where  $q = p + (\nabla \varphi(x'), 0)$  and  $\phi(y') = \varphi(y') - \varphi(x') - \nabla \varphi(x') \cdot (y' - x')$  which satisfies  $\phi(x') = \partial_i \phi(x') = 0$  for  $i = 1, \dots, N-1$ .

So the result obtained for  $v$  would transfer to  $u$  replacing  $p$  with  $q$ .

We can start. For  $k = 0$ , taking  $p_o = 0$  yields the desired inequality. Suppose that  $p_k$  has been constructed.

Choose  $\beta$  small enough in order that  $\rho^\beta > \frac{1}{2}$ .

With the above assumption on  $\varphi$ , let  $\varphi_k(y') = r_k^{-1-\beta} \varphi(x' + r_k(y' - x'))$ , which satisfies, for  $\beta < \beta_o$ ,  $|\varphi_k|_{C^{1,\beta}(B_1(x'))} \leq |\varphi|_{C^{1,\beta}}$ .

We consider

$$u_k(y) = r_k^{-1-\beta} (u(r_k(y - x) + x) - p_k \cdot (r_k(y - x) + x)).$$

$u_k$  is well defined on  $B_1(x) \cap \{y_N > a_k(y')\}$ , where  $a_k(y') = x_N(1 - \frac{1}{r_k}) + \frac{a(r_k(y' - x') + x')}{r_k}$ . Note that on  $B_1(x) \cap \{y_N > a_k(y')\}$ ,  $|x_N| < r_k(1 + \|D^2 a\|_\infty)$ .

It is immediate to see that  $u_k$  is a solution of

$$\begin{cases} |p_k r_k^{-\beta} + \nabla u_k|^\alpha (F(D^2 u_k) + h_k \cdot (p_k r_k^{-\beta} + \nabla u_k)) = f_k & \text{in } B_1(x) \cap \{y_N > a_k(y')\} \\ u_k + p_k r_k^{-\beta} \cdot y = p_k r_k^{-\beta} (r_k - 1) \cdot x + \varphi_k(y') & \text{on } B_1(x) \cap \{y_N = a_k(y')\} \end{cases}$$

with  $f_k(y) = r_k^{1-\beta(1+\alpha)} f(r_k(y - x) + x)$  and  $h_k(y) = r_k h(r_k(y - x) + x)$ .

We now prove that  $a_k$  satisfies  $\|a_k\|_{C^2} \leq C$  for some constant  $C$  which does not depend on  $k$ . Indeed, in  $B_1(x) \cap \{y_N > a_k(y')\}$ ,  $r_k \geq \frac{|x_N|}{1 + \|D^2 a\|_\infty}$ , which implies that

$$|a_k(y')| \leq |x_N| + (1 + \|D^2 a\|_\infty) + \frac{r_k^2 \|D^2 a\|_\infty}{r_k} \leq \text{diam} \Omega + 2(1 + \|D^2 a\|_\infty)$$

and  $|\nabla a_k| = |\nabla a|$ , finally  $\|D^2 a_k\| = r_k \|D^2 a\|_\infty \leq \|D^2 a\|_\infty$ .

Furthermore, as long as  $\beta < \frac{1}{1+\alpha}$ ,

$$\text{osc}_{B_1(x) \cap \{y_N > a_k(y')\}} u_k \leq 1, \quad \|f_k\|_{L^\infty(B_1(x) \cap \{y_N > a_k(y')\})} \leq \epsilon_o \quad \text{and} \quad \|h_k\|_\infty \leq \|h\|_\infty.$$

Hence, using Lemma 3.3 with obvious changes, there exists  $q_{k+1} \in \mathbb{R}^N$  such that

$$\text{osc}_{B_\rho(x) \cap \{y_N > a_k(y')\}}(u_k(y) - q_{k+1} \cdot y) \leq \frac{\rho}{2}.$$

Defining  $p_{k+1} = p_k + q_{k+1}r_k^\beta$ , with the assumptions on  $\beta$  and  $\rho$ , one gets:

$$\text{osc}_{B_{r_{k+1}}(x) \cap \{y_N > a(y')\}}(u(y) - p_{k+1} \cdot y) \leq \frac{\rho}{2}r_k^{1+\beta} \leq r_{k+1}^{1+\beta},$$

since the oscillation is invariant by translation. This is the desired conclusion.

There remains to prove the flatness lemmata. We start by the interior case with lower order terms.

*Proof of Lemma 3.2.* Suppose by contradiction that there exist a sequence of functions  $(f_n)_n$  whose norm go to zero, a sequence of  $(p_n)_n \in \mathbb{R}^N$  and a sequence of functions  $(u_n)_n$  with  $\text{osc } u_n \leq 1$ , solutions of

$$|p_n + \nabla u_n|^\alpha (F(D^2 u_n) + h(y) \cdot (\nabla u_n + p_n)) = f_n, \quad (3.2)$$

such that, for all  $q \in \mathbb{R}^N$ ,

$$\text{osc}_{B_\rho}(u_n(y) - q \cdot y) > \frac{\rho}{2}. \quad (3.3)$$

Let us suppose first that  $(p_n)_n$  is bounded so, then up to subsequence, it converges to  $p_\infty$ . Considering  $v_n(y) = u_n(y) + p_n \cdot y$  and using the compactness Remark 2.6, we can extract from  $(v_n)_n$  a subsequence converging to a limit  $v_\infty$ , which satisfies

$$|\nabla v_\infty|^\alpha (F(D^2 v_\infty) + h(x) \cdot \nabla v_\infty) = 0.$$

Remark next that the solutions of such an equations are solutions of

$$F(D^2 v_\infty) + h(\cdot) \cdot \nabla v_\infty = 0.$$

as it is the case for  $h = 0$  (see [11]). But, passing to the limit in (3.3) gives that  $\text{osc}_{B_\rho}(v_\infty - (q - p_\infty) \cdot x) > \frac{\rho}{2}$ . This contradicts the regularity results known for solutions of this equation (see [14]) and it ends the case where the sequence  $(p_n)_n$  is bounded.

In the case where  $(p_n)$  is unbounded, take a subsequence such that  $\frac{p_n}{|p_n|}$  converges to some  $p_\infty$ .



**Claim:** There exist  $q_\infty \in \mathbb{R}^N$  and a subsequence  $\sigma(n)$  such that

$$\lim_{n \rightarrow \infty} h(y) \cdot p_{\sigma(n)} = h(y) \cdot q_\infty$$

uniformly in  $B_r$  for any  $r < 1$ .

We postpone the proof of that claim and end the proof of Lemma 3.2.

We now divide the equation (3.2) by  $|p_n|^\alpha$  and get, with  $e_n = \frac{p_n}{|p_n|}$  and  $a_n = \frac{1}{|p_n|}$ ,

$$|a_n \nabla u_n + e_n|^\alpha (F(D^2 u_n) + h(y) \cdot (\nabla u_n + p_n)) = \frac{f_n}{|p_n|^\alpha}.$$

Using Remark 2.11 and the claim, a subsequence of  $u_{\sigma(n)}$  converges to  $u_\infty$  a solution of the limit equation

$$F(D^2 u_\infty) + h(y) \cdot (\nabla u_\infty + q_\infty) = 0.$$

On the other hand,  $\text{osc}(u_\infty - q' \cdot x) > \frac{1}{2}\rho$ , which contradicts the regularity of solutions of such equations.

**Proof of the Claim.** Let  $V \subset \mathbb{R}^N$  be the space generated by  $h(B_r)$ . Let  $q_n = \Pi_V p_n$  be the projection of  $p_n$  on  $V$ , hence  $h(x) \cdot p_n = h(x) \cdot q_n$ .

Suppose by contradiction that the sequence  $(q_n)_n$  goes to infinity in norm. There exists a subsequence  $q_{\sigma(n)}$  such that  $\frac{q_{\sigma(n)}}{|q_{\sigma(n)}|} \rightarrow \bar{q}$ , for some  $\bar{q}$  such that  $|\bar{q}| = 1$  and  $\bar{q} \in V$ , furthermore we have that  $\frac{p_{\sigma(n)}}{|p_{\sigma(n)}|} \rightarrow \bar{p}$  such that  $|\bar{p}| = 1$  and  $\frac{|q_{\sigma(n)}|}{|p_{\sigma(n)}|} \rightarrow \tilde{p} \in \mathbb{R}$ .

We divide the equation by  $|p_n|^\alpha |q_n|$  and observe that the functions  $v_n = \frac{u_n}{|q_n|}$  satisfy

$$\left| \frac{p_n}{|p_n|} + \frac{|q_n|}{|p_n|} \nabla v_n \right|^\alpha \left( F(D^2 v_n) + \frac{h(y) \cdot q_n}{|q_n|} + h(y) \cdot \nabla v_n \right) = \frac{f_n}{|p_n|^\alpha |q_n|}.$$

And the sequence  $v_n$  converges to zero.

Using the compactness of  $(v_{\sigma(n)})_n$ , see Remark 2.11, one gets that it converges to a solution of

$$(\bar{p} + \tilde{p} \nabla v_\infty)^\alpha (F(D^2 v_\infty) + h(y) \cdot \bar{q} + h(y) \cdot \nabla v_\infty) = 0$$

but, since  $v_\infty = 0$  we get

$$h(y) \cdot \bar{q} = 0 \quad \text{for all } y \in B_r.$$

This means that  $\bar{q}$  is both in  $V$  and  $V^\perp$ , hence  $\bar{q} = 0$ , which is a contradiction and the sequence  $(q_n)_n$  is bounded. This gives the claim, indeed, up to a subsequence  $q_n$  converges to  $q_\infty$  and

$$\lim_{n \rightarrow \infty} h(y) \cdot p_{\sigma(n)} = \lim_{n \rightarrow \infty} h(y) \cdot q_{\sigma(n)} = h(y) \cdot q_\infty.$$

This ends the proof.

We now want to prove Lemma 3.3. In the case of a non straight boundary it requires the following technical proposition, whose proof is postponed until the end of the section.

**Proposition 3.6** *Suppose that  $a$  is not identically zero in  $B'_\delta(0)$  and  $a(0) = \nabla a(0) = 0$ . Suppose that  $(p_n)_n$  is a sequence in  $\mathbb{R}^N$  such that, for all  $n$  and for all  $x \in B'_\delta(0)$ ,*

$$|p_n \cdot (x', a(x'))| \leq C$$

*for some constant  $C$ . Then  $(p_n)_n$  is a bounded sequence.*

*Proof of Lemma 3.3.* We assume first that  $h = 0$ .

Note that if  $B_1(x) \cap \{y_N = a(y')\} = \emptyset$  then it is sufficient to use the result of [11]. So we now assume that  $B_1(x) \cap \{y_N = a(y')\} \neq \emptyset$ .

Let  $\rho$  and  $q = q_{x,\rho}$  be so that any  $u$ , solution of

$$\begin{cases} F(D^2u) + h(y) \cdot \nabla u = 0 & \text{in } B_1(x) \cap \{y_N > a(y')\} \\ u = \varphi & \text{on } B_1(x) \cap \{y_N = a(y')\}, \end{cases}$$

satisfies  $\text{osc}_{B_\rho(x)}(u(y) - q \cdot y) \leq \frac{\rho}{2}$ .

We argue by contradiction and suppose that for all  $n$ , there exist  $x_n \in \overline{B}$  and  $p_n \in \mathbb{R}^N$ ,  $|f_n|_{L^\infty(\Omega)} \leq \frac{1}{n}$  and a function  $u_n$  with  $\text{osc}(u_n) \leq 1$  solution of

$$\begin{cases} |p_n + \nabla u_n|^\alpha F(D^2u_n) = f_n & \text{in } B \cap \{y_N > a(y')\} \\ u_n(y) + p_n \cdot y = \varphi(y') & \text{on } B \cap \{y_N = a(y')\} \end{cases} \quad (3.4)$$

such that

$$\text{osc}_{B_\rho(x_n)}(u_n(y) - q \cdot y) > \frac{\rho}{2}. \quad (3.5)$$

Extract from  $(x_n)_n$  a subsequence which converges to  $x_\infty \in \overline{B} \cap \{y_N \geq a(y')\}$ .

We denote in the sequel  $B_\infty = \cap_{n \geq N_1} B_1(x_n)$  which contains  $B_\rho(x_\infty)$  as soon as  $N_1$  is large enough.

**The case where  $T$  is not straight.** Observing that  $u_n - u_n(x_n)$  satisfies the same equation as  $u_n$ , has oscillation 1 and is bounded, we can suppose that the sequence  $(u_n)$  is bounded. This, together with the boundary condition and Proposition 3.6, implies that  $(p_n)$  is bounded.

So, up to subsequences,  $u_n$  converges to some  $u_\infty$ , uniformly on  $\overline{B_r} \cap \{y_N \geq a(y')\}$  for every  $B_r \subset\subset B_\infty$ , (due to Corollary 2.5), and  $p_n$  converges to  $p_\infty$ . Furthermore,  $(u_\infty, p_\infty)$  solves

$$\begin{cases} |p_\infty + \nabla u_\infty|^\alpha F(D^2 u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > a(y')\} \\ u_\infty + p_\infty \cdot y = \varphi(y) & \text{on } B_\infty \cap \{y_N = a(y')\}. \end{cases}$$

Using Lemma 6 in [11] one gets that

$$\begin{cases} F(D^2 u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > a(y')\} \\ u_\infty + p_\infty \cdot y = \varphi(y) & \text{on } B_\infty \cap \{y_N = a(y')\}. \end{cases}$$

On the other hand, by passing to the limit in (3.5), one obtains

$$\text{osc}_{B_\rho(x_\infty) \cap \{y_N > 0\}}(u_\infty(y) - q \cdot y) \geq \frac{\rho}{2}.$$

This is a contradiction with the assumption on  $\rho$  and  $q$  and it ends the proof.

**The case where  $T = \{y_N = 0\}$ .** Let  $p_n = p'_n + p_n^N e_N$ . The boundedness of  $u_n$  and the boundary condition imply that  $(p'_n)_n$  is bounded. If  $(p_n^N)_n$  is bounded just proceed as above. So we suppose that  $p_n^N$  is unbounded.

Dividing (3.4) by  $|p_n^N|^\alpha$  it becomes

$$\begin{cases} |\frac{p_n}{p_n^N} + \frac{1}{p_n^N} \nabla u_n|^\alpha F(D^2 u_n) = \frac{f_n}{|p_n^N|^\alpha} & \text{in } B_\infty \cap \{y_N > 0\} \\ u_n + p'_n \cdot y = \varphi(y') & \text{on } B_\infty \cap \{y_N = 0\} \end{cases}$$

Denoting by  $p'$  the limit of a subsequence of  $p'_n$ , and  $u_\infty$  the limit of a subsequence of  $(u_n)$ , one gets by passing to the limit and using Corollary 2.10 that

$$\begin{cases} F(D^2 u_\infty) = 0 & \text{in } B_\infty \cap \{y_N > 0\} \\ u_\infty + p' \cdot y = \varphi(y') & \text{on } B_\infty \cap \{y_N = 0\}. \end{cases}$$

Passing to the limit in (3.5) one gets that  $\text{osc}_{B_\rho(x_\infty)}(u_\infty - q \cdot y) > \frac{\rho}{2}$ , a contradiction. This ends the proof for  $h = 0$ .

We briefly point out the differences in the case  $h \neq 0$ . It is sufficient to treat the case of a straight boundary, the other cases being as before. Indeed, we already know that if the boundary is not locally straight,  $(\vec{p}_n)_n$  is bounded.

In the case where the boundary is locally straight, say  $T = \{y_N = 0\}$  so that the only possibly unbounded component is  $p_n \cdot e_N$ .

The Claim in the interior flatness Lemma 3.2 implies that for any open set  $D \subset B_r(x) \cap \{y_N > a(y')\}$ ,  $h(y) \cdot e_N = 0$  for any  $y \in D$ . By the arbitrariness of  $D$ , this implies that  $h(y) \cdot e_N = 0$  in  $B_r(x) \cap \{y_N > a(y')\}$ . We can now apply Proposition 2.7 and end as in the case  $h = 0$ .

We now want to give the  
*proof of Proposition 3.6.*

Since  $a$  is not identically 0, and  $a(0) = 0$ ,  $\nabla a(0) = 0$ , there exists  $i$  such that  $\partial_i a$  is not identically zero in  $B'_\delta(0)$  for some  $\delta > 0$ .

Hence, there exist  $y_1^i \in B'_\delta(0)$  and  $y_2^i \in B'_\delta(0)$  such that

$$y_1^i e_i + a(y_1^i e_i) e_N \quad \text{and} \quad y_2^i e_i + a(y_2^i e_i) e_N$$

are linearly independent. Indeed, assume by contradiction that, for  $y_o \neq 0$  fixed in  $B'_\delta(0)$ , for all  $y \in B'_\delta(0)$ , there exists  $t \in \mathbb{R}$  such that  $(ye_i, a(ye_i)) = t(y_o e_i, a(y_o e_i))$ . Then,  $a(ye_i) = \frac{a(y_o e_i)y}{y_o}$ , which implies that  $a_{,i} = cte$  in  $B'_\delta(0)$  which is a contradiction.

Defining  $E_j^i = y_j^i e_i + a(y_j^i) e_N$ , for  $i \in [1, N-1]$  and  $j = 1, 2$  one gets

$$|(p_n^i, p_n^N) \cdot (E_j^i)| \leq C \text{ for } j = 1, 2.$$

Since  $E_j^i$  are two linearly independent vectors, it implies that  $(|p_n^i| + |p_n^N|)_n$  is bounded. Suppose now that  $a_{,i} \equiv 0$ , then replacing  $E_j^i$  by  $e_i$  in the previous reasoning, one obtains that  $|p_n^i|$  is bounded. Since  $a$  is not identically zero, there exists at least one  $i$  for which the first situation occurs, so we get that for all  $j$ ,  $(p_n^j)_n$  is bounded.

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